The Logistic Differential Equation

Goals

- MATH: To analyze the behavior of solutions of an ordinary differential equation geometrically
- MATH: To analyze stability behavior of equilibria of an ordinary differential equation geometrically and symbolically
- MATH: To introduce a basic numerical technique for approximation solutions to differential equations.
- BIO: To explore the logistic model, and variations caused by introducing harvesting
- COMP: To use MAPLE to analyze an ordinary differential equation

Computational Tools: MAPLE

Let $N = N(t)$ denote the size of a population at time t. The logistic differential equation is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where r and K are positive constants. The parameter $r > 0$ is a growth parameter, and the parameter $K > 0$ denotes the carrying capacity for the population. This population model is an example of a continuous, density dependent population model. The model is density dependent, because the number of offspring per individual is dependent on current population size. In this activity, we will explore geometric and numerical techniques for studying solutions of this differential equation.

1. Open a blank Maple worksheet. Since we will be using differential equation commands, we need to load the differential equation tools package *DETools*. To do this, type the command:

   with(DETools):

2. We will next enter the logistic equation in terms of the parameters r and K by typing the command:

   *Logistic := (r, K) → diff(N(t), t) = r*N(t)*(1-N(t)/K);*

   We can verify that this gives us what we desire by observing that the command

   Logistic(0.1,100);

   returns the differential equation

   $$\frac{d}{dt}N(t) = 0.1N(t)\left(1 - \frac{1}{100}N(t)\right).$$

3. Our first task is to create a direction field (slope field) plot for this differential equation with $r = 0.1$ and $K = 100$. A direction field plot enables us to observe the behavior of a family of solutions at once. To accomplish this, enter the command:

   DEplot(Logistic(0.1, 100), N(t), t = 0 .. 40, N = 0 .. 120);

   We can observe solutions on this direction field plot by plotting several solutions on the direction field. To plot solutions with initial conditions $N(0) = 0$, $N(0) = 10$, $N(0) = 60$, $N(0) = 100$, $N(0) = 120$, we modify the previous command to

   DEplot(Logistic(0.1, 100), N(t), t = 0 .. 40, [[N(0) =0],[N(0)=10],[N(0) = 60],[N(0) = 100],[N(0)=120]],N = 0 .. 120);

4. Print out the direction field plot containing the five particular solution curves from the preceding step (you would do this at the end of your lab event, or export to a bitmap or jpeg, and print those). Label each of these solution curves with its initial condition. In Maple, insert a paragraph (Insert Paragraph...) and write some text to describe the behavior of each of these solutions (in terms of monotonicity, concavity, and long-term behavior). Which of these solutions appear to be equilibrium solutions? What role does the carrying capacity play in these solutions?

5. Modify the previous command to consider other possible initial conditions. It seems that the initial value $N(0)$ and its relationship to the carrying capacity K have some relevance to the shape of the solution curve. Conjecture what you think the relationship of $N(0)$ to the solution curve's behavior might be. In particular, describe what happens in the five cases with $N(0) = 0$, $0 < N(0) < K/2$, $K/2 < N(0) < K$, $N(0) = K$, and $N(0) > K$. Use a text paragraph (Insert Paragraph...) to add this information to your Maple worksheet.

6. We now determine the equilibria of the logistic differential equation. The equilbria are determined by setting the right hand side of the differential equation equal to zero and then solving for N. The following two commands define dN/dt as a function of N and then solves for the equilibria:

   *DerivPlot := (r, K) → r*N*(1 - N/K);*
   try this in math mode: $DerivPlot := (r, K) \rightarrow r * N * (1 - N/K)$;
   solve(DerivPlot(r, K) = 0, N);

7. These equilibria can be observed graphically as well by considering a plot of dN/dt versus N. The equilibrium values of N are the values of N for which the plot of dN/dt intersects the N-axis. We create this plot with the command:

   plot(DerivPlot(.1, 100), N = -10 .. 120, labels = [N, dN/dt]);

   Print out this plot. Place large dots on the equilibrium values of N on the N-axis of this plot. From this plot, we can determine the values of N at which solution curves increasing by observing where $dN/dt > 0$. For each interval on which solutions are increasing, mark the interval on the N-axis with an arrow pointing to the right. Similarly, solutions are decreasing when $dN/dt < 0$. Mark each interval for which solutions are decreasing with an arrow pointing to the left. The N-axis along with the labeled equilibria and arrows indicating increasing and decreasing behavior is called the *phase line* for the differential equation. As labeled, the phase line encodes the key dynamic behavior of the differential equation.

8. The phase line also encodes the stability of each equilibrium value. If the arrows on both sides of an equilibrium point toward the equilibrium value, the equilibrium is *locally stable*. If the arrows on both sides of an equilibrium point away from the equilibrium value, the equilibrium is *unstable*. It is possible that the arrows on either side of an equilibrium point in the same direction. This indicates that solutions approach the equilibrium from one side and diverge from the equilibrium on the other side. In this case, the equilibrium is called *semistable*. Using the phase line, determine the stability of each equilibrium value. Notice that your answers do not depend on the particular values of r and K? To formulate your answer in general (for arbitrary values of $r > 0$ and $K > 0$), consider the original differential equation.

9. By looking at the plot of dN/dt versus N, we can also observe the concavity of solution curves by noting where dN/dt is increasing and decreasing. We will use Maple to calculuate the second derivative of N directly. The command

   Deriv := (t) → Logistic(r, K);

   defines the right hand side of the differential equation as function of t. The command *diff(Deriv(t), t);*

computes the second derivative of N. We will now manipulate the right hand side of the resulting equation to easily determine points of inflection and intervals for with solutions are concave up or concave down. Highlight the right hand side of this equation and copy it to a new command line. Before typing Enter, replace each occurrence of $\frac{d}{dt}N(t)$ with $rN\left(1-\frac{N}{K}\right)$. Add a semicolon at the end of this expression and type Enter. The result is an expression for $\frac{d^2N}{dt^2}$ in terms of $N(t)$, r, and K. We can factor this expression with the command:

factor(%);

(The "%" symbol recalls the expression on the preceding line in the Maple Worksheet.) Now using this expression, precisely determine values of N for which points of inflection occur as well as the concavity behavior of the solutions. Compare your results with the plot of dN/dt versus N.

10. There is an analytic criterion to determine local stability of an equilibrium of an autonomous ordinary differential equation. The criterion can be stated as follows. Let $dN/dt = f(N)$ be a differential equation for equation f is a differential function of N, and let $\hat{N}$ be an equilibrium value. Then if $f'(\hat{N}) < 0$, $\hat{N}$ is locally stable, and if $f'(\hat{N}) > 0$, the equilibrium $\hat{N}$ is unstable. Use this criterion to determine the stability of each of the equilibrium values of the logistic differential equation. Compare your results with your phase line analysis. (Your results should be the same!)

11. The solution curves plotted in a previous step we calculuated using a numerical approximation technique for solving differential equations. We will briefly explore a simple numerical technique called *Euler's Method*, which depends on the basic calculus concept of a tangent line approximation. Euler's Method uses the idea of stringing together tangent line segments to create an approximate solution. To review the concept of tangent line approximation, we will plot the tangent line to the solution of this differential equation (again with $r = 0.1$ and $K = 100$) with initial condition $N(0) = 10$. Once you have determined the equation of this tangent line, define a function, Y, to represent this function in Maple. We use the following sequence of commands to plot the tangent line on the direction field plot:

   with(plots);
   plot1:=DEplot(Logistic(0.1, 100), N(t), t = 0 .. 40, N = 0 .. 120):
   plot2:=plot(Y(t),t=0..40,N=0..120):
   plots[display]({plot1,plot2});

   (*Euler's Method*) Euler's Method is usually computed using a small stepsize for the independent variable. This means that the tangent line segments that we string together are short. The shorter these segments are, the better the approximate solution is. To illustrate how Euler's Method appears after several steps on our direction field plot, we will use a large stepsize to illustrate the process. Enter the following two commands:

   DEplot(Logistic(.1, 100), N(t), t = 0 .. 40, [[N(0) =0],[N(0)=10],[N(0) = 60],[N(0) = 100],[N(0)=120]], N = 0 .. 120, method = classical[foreuler], stepsize = 5);

   DEplot(Logistic(.1, 100), N(t), t = 0 .. 40, [[N(0) =0],[N(0)=10],[N(0) = 60],[N(0) = 100],[N(0)=120]], N = 0 .. 120, method = classical[foreuler], stepsize = 1);

   Notice that the approximate solutions obtained with a stepsize of 1 are closer to the solution curves obtained above that the approximate solutions obtained with a stepsize of 5. Euler's Method is not used much in practice, but it does illustrate the basic idea behind numerically approximating solutions to differential equations.

12. *(Conclusion)* To formulate your conclusion for this activity, describe the long term behavior of a population that behaves according to the logistic differential for arbitrary initial populations, $N(0)$. When are population trends increasing? When are these populations increasing most rapidly? When are population trends increasing? What role does the carrying capacity play in this long term behavior?

References

[1] L. Edelstein-Keshet, *Mathematical Models in Biology*, Classics in Applied Mathematics **46**, SIAM, 2005.

[2] F. Garvan, *The Maple Book*, Chapman & Hall/CRC, 2002.

[3] C. Neuhauser, *Calculus for Biology and Medicine, 2e*, Pearson Prentice Hall, 2004.