MATH 221 Biocalculus II
Project 6
Nonlinear Systems of Differential Equations II

Goals

- To explore the Lotka-Volterra Predator-Prey Model
- To explore a model for microbal growth

Computational Tools: Maple

The discussion in this project follows that of Section 11.4 of the text by Neuhauser [2].

Lotka-Volterra Predator-Prey Model

The Lotka-Volterra Predator-Prey Model is the most basic predator-prey model with coupled oscillations. (See [1], [3].) We will use t as our time variable, and let $t = 0$ denote the time at which the two populations are first measured. (We are suppressing our time units here.) The population size of the prey at time t is denoted by $N(t)$, and the population size of the predator at time t is denoted by $P(t)$. We make the following assumptions:

1. The prey species will grow exponentially in the absence of the predatory species, with an intrinsic growth rate of r.
2. The predatory species will die off if the prey species is absent.
3. The species can coexist.

We further assume that frequency of attacks obeys the law of mass action. That is, the numbers of attacks (deaths) of the prey species is proportional to the production PN. We let $a > 0$ denote this proportionality constant, which is called the *attack rate*. Thus, N decreases by aPN. As the predatory species depends on the prey to survive, we need to make an assumption about how each kill converts into new offspring for the predatory species. We assume that there are $b > 0$ new offspring per each attack. That is, the predatory species increases by $baPN$. Finally, we assume that the predatory species dies at a rate proportional to its population size.

1. Write down the system of differential equations described above.
2. What are the units on the parameter a?
3. In the $NP-$plane, plot the nullclines and find the equilibria.
4. Find the Jacobian matrix for the system.
5. Determine the eigenvalues at each equilibrium point. Analyze the stability from the eigenvalue information if you can. (Recall the conditions when linerization can be used to determined stability.)
6. There is one equilibrium $(\hat{N}, \hat{P})$ for which we cannot use the linearization. For this equilibrium, we can use separation of variables to find an equation relating to P and N. To begin this process, we use the following equation, which follows from the chain rule:

$$\frac{dP}{dN} = \frac{dP/dt}{dN/dt} \tag{1}$$

Using this equation, write down an expression for dP/dN in terms of $N, P, a, b, d,$ and r. Then use separation of variables to find an equation for relating N and P of the form

$$f(N,P) = K, \tag{2}$$

where K is a constant.

7. Show that the function f achieves its absolute maximum $K_{\max}$ at $(\hat{N}, \hat{P})$.

8. Consider the Lotka-Volterra predator-prey model:

$$\begin{aligned}\frac{dN}{dt} &= 5N - PN \\ \frac{dP}{dt} &= PN - P\end{aligned}$$

   Plot solution curves in the $NP-$plane for at least 5 values of $K \leq K_{\max}$.

9. Using the same problem, plot the curves $N = N(t)$ and $P = P(t)$ in the same plane for several different initial conditions. (These last two steps can be done in Berkeley Madonna with a single program!)

10. (*based on problems in Neuhauser [2]*) Consider the following modification of the Lotka-Volterra model that assumes that the prey population evolves logistically:

$$\begin{aligned}\frac{dN}{dt} &= aN\left(1 - \frac{N}{K}\right) - bPN \\ \frac{dP}{dt} &= cPN - dP\end{aligned}$$

   (a) Determine the conditions on the parameters for which an equilibrium exists where both N and P are positive. Is there a minimum carrying capacity K for such an equilibrium to exist?

   (b) Show that the equilibrium you found in part (a) is locally stable.

   (c) Analyze what happens to the predator and prey populations at equilibrium when each of the parameters a, b, c, and K are increased (separately).

   (d) Plot solutions of this system in the NP-plane in the case that $a = 1$, $b = 4$, $c = 1$, $d = 5$, and $K = 10$.

References

[1] A.J. Lotka, "The growth of mixed populations: two species competing for a common food supply," *Journal of the Washington Academy of Sciences* **22**, 461-469, 1932.

[2] C. Neuhauser, *Calculus for Biology and Medicine, 2e*, Pearson Prentice Hall, 2004.

[3] V. Volterra, "Fluctuation in the abundance of species considered mathematically," *Nature* **118**, 558-560, 1926.